$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ $E_b = \sigma T^4$

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$e^{i\pi} = -1$ $\frac{u}{u_\tau} = \frac{1}{\kappa}\ln\frac{yu_\tau}{\nu} + C$

THEORY GUIDE

# Subsoil Temperature Web Application

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# 1 Introduction

You can find the Atkinson Science Subsoil Temperature web application at the web address https://atkinsonscience.co.uk/WebApps/Construction/SubsoilTemperature.aspx. There is a user guide that you can download at the same address.

The Subsoil Temperature web application calculates the variation in temperature with time and depth for a given subsoil when the air temperature, and therefore the surface temperature, is sinusoidal. Variations in temperature are assumed to be significant only in the direction normal to the ground. Given the latter assumption, we can represent the temperature variation in the subsoil by the time-dependent, one-dimensional differential equation of heat conduction. When the thermal properties of the subsoil are non-uniform, the equation is written

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \qquad (1)$$

where $T$ is temperature, $t$ is time, $x$ is depth and $\rho$, $c$ and $k$ are the thermal properties density, specific heat capacity and thermal conductivity. We assume that the thermal properties are uniform. Eqn. (1) then becomes

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad (2)$$

where $\alpha = k/(\rho c)$ is the (constant) thermal diffusivity of the subsoil. The thermal diffusivity cannot be negative, so we can write $\alpha = h^2$ and

$$\frac{\partial T}{\partial t} = h^2 \frac{\partial^2 T}{\partial x^2} \qquad (3)$$

# 2 Boundary value problem

To obtain the variation with time and depth in the subsoil when the surface temperature varies sinusoidally with time we must solve the boundary value problem:

$$\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2} \qquad (4)$$

$$x = 0, \quad u = T - T_{mean} = T_{amp} \sin \omega t \qquad (5)$$

$$x = \infty, \quad u \neq \infty \qquad (6)$$

where $T_{mean}$ and $T_{amp}$ are the temperature mean and amplitude at the surface.

# 3 General solution of the heat conduction equation

First we must find the general solution of the heat conduction equation (4). Eqn. (4) is a homogeneous linear partial differential equation of order 2 with constant coefficients. The order of the equation is 2, so we must find two arbitrary independent functions which satisfy the equation identically. Since the equation is homogeneous and linear with constant coefficients, we can obtain the general solution by assuming $u = e^{at+bx}$. Substituting into (4) gives

$$(a - h^2b^2)e^{at+bx} = 0$$

or

$$a - h^2b^2 = 0$$

Rearranging this equation gives

$$b = \pm\sqrt{\frac{a}{h^2}}$$

Thus we have the general solution

$$u = Ae^{at\pm\sqrt{a/h^2}x} \qquad (6)$$

# 4 Solution of the boundary value problem

If we replace $a$ in Eqn. (6) by $\pm i\gamma$ then we have

$$T = A\exp\left(\pm i\gamma t \pm x\sqrt{\frac{\gamma}{h^2}}\sqrt{\pm i}\right) \qquad (7)$$

If we introduce the identities

$$\sqrt{i} = \pm\frac{(1+i)}{\sqrt{2}}$$

and

$$\sqrt{-i} = \pm\frac{(1-i)}{\sqrt{2}}$$

then (7) becomes

$$u = A\exp\left(\pm i\gamma t \pm x\sqrt{\frac{\gamma}{2h^2}}(1 \pm i)\right)$$

or

$$u = A\exp\left(\pm x\sqrt{\frac{\gamma}{2h^2}}\right)\exp\left(\pm i\left(\gamma t \pm x\sqrt{\frac{\gamma}{2h^2}}\right)\right)$$

The boundary condition (6) rules out the terms in

$$+x\sqrt{\frac{y}{2h^2}}$$

so we are left with

$$u = A\exp\left(-x\sqrt{\frac{\gamma}{2h^2}}\right)\exp\left(\pm i\left(\gamma t - x\sqrt{\frac{\gamma}{2h^2}}\right)\right)$$

We can take the two possible solutions

$$u = A\exp\left(-x\sqrt{\frac{\gamma}{2h^2}}\right)\exp\left(i\left(\gamma t - x\sqrt{\frac{\gamma}{2h^2}}\right)\right)$$

and

$$u = A\exp\left(-x\sqrt{\frac{\gamma}{2h^2}}\right)\exp\left(-i\left(\gamma t - x\sqrt{\frac{\gamma}{2h^2}}\right)\right)$$

and subtract one from the other to obtain

$$u = A\exp\left(-x\sqrt{\frac{\gamma}{2h^2}}\right)\left[\exp\left(i\left(\gamma t - x\sqrt{\frac{\gamma}{2h^2}}\right)\right) - \exp\left(-i\left(\gamma t - x\sqrt{\frac{\gamma}{2h^2}}\right)\right)\right]$$

This equation can be simplified to

$$u = B\exp\left(-x\sqrt{\frac{\gamma}{2h^2}}\right)\sin\left(\gamma t - x\sqrt{\frac{\gamma}{2h^2}}\right) \qquad (8)$$

Applying boundary condition (5) gives

$$B\sin\gamma t = T_{amp}\sin\omega t$$

so $B = T_{amp}$ and $\gamma = \omega$.

Substituting into (8) gives

$$u = T_{amp}\exp\left(-x\sqrt{\frac{\omega}{2h^2}}\right)\sin\left(\omega t - x\sqrt{\frac{\omega}{2h^2}}\right) \qquad (9)$$

Substituting $u = T - T_{mean}$ and $h^2 = \alpha$ into (9) gives the solution of the boundary value problem

$$T = T_{mean} + T_{amp}\exp\left(-x\sqrt{\frac{\omega}{2\alpha}}\right)\sin\left(\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right) \qquad (10)$$

If the time $t$ is measured in days then the period $P$ of the surface temperature will be 365 days and the angular frequency $\omega$ will be

$$\omega = \frac{2\pi}{P} = \frac{2\pi}{365} = 0.0172\ \text{day}^{-1}$$

In addition, if the depth $x$ is measured in metres then the units of thermal diffusivity of the subsoil $\alpha$ must be m$^2$ day$^{-1}$.

As the depth $x$ increases, the exponential function in the second term on the right of Eqn. (10) will cause this term to tend to zero and the subsoil temperature to tend to $T_{mean}$.

The term $-x\sqrt{(\omega/2\alpha)}$ in the sine function of Eqn. (10) represents the phase lag between the temperature in the subsoil and the temperature at the surface. When this term is equal to $2\pi$ the temperature of the subsoil lags behind the surface temperature by one year. The depth at which the temperature of the subsoil lags by one year is

$$x[\mathrm{m}] = \frac{2\pi}{\sqrt{\frac{\omega}{2\alpha}}} = \frac{2\pi}{\sqrt{\frac{1}{2\alpha} \times \frac{2\pi}{P}}} = 2\sqrt{\pi\alpha P} = 2\sqrt{\pi\alpha\, 365} = 67.73\sqrt{\alpha} \qquad (11)$$

At this depth the amplitude of the subsoil temperature is so small that engineers often take it to be the depth at which the temperature becomes constant.